

# Informational Efficiency under Short Sale Constraints\*

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## Abstract

A constrained informationally efficient market is defined to be one whose price process arises as the outcome of some equilibrium where agents face restrictions on trade. This paper investigates the case of short sale constraints, a setting which despite its simplicity, generates new insights. In particular, it is shown that short sale constrained informationally efficient markets always admit equivalent supermartingale measures and local martingale deflators, but not necessarily local martingale measures. And if in addition some local martingale deflator turns the price process into a true martingale, then the market is constrained informationally efficient. Examples are given to illustrate the subtle phenomena that can arise in the presence of short sale constraints, with particular attention to representative agent equilibria and the different notions of no arbitrage.

KEY WORDS: informational efficiency, short sales, no arbitrage, no dominance, equilibrium, representative agents, martingales measures, local martingales, supermartingales, local martingale deflators.

JEL CLASSIFICATION: G11, G12, G14, D52, D53

## 1 Introduction

Since Fama [11] published his seminal paper on market efficiency four decades ago, a vast body of empirical research has emerged evaluating the informational efficiency of real-world markets with mixed results. These mixed results are partly due to the the notorious *joint model hypothesis* problem—to test for informational efficiency one must jointly hypothesize an equilibrium model. Competing theories to the rational competitive markets paradigm

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have also been proposed to reconcile this conflicting evidence such as the behavioral approach to finance and economics (excellent surveys include Hirshleifer [15], Shleifer [36], Barberis and Thaler [1]) and the adaptive market hypothesis (see, for instance, Lo [28]). In all cases, informational efficiency, i.e. the degree to which information about fundamentals is quickly and accurately reflected in asset prices, has proved to be an immensely powerful tool for understanding the behavior of asset markets.

To circumvent this joint model hypothesis, Jarrow and Larsson [17] formalized the definition of informational efficiency, consistent with Fama's original ideas, as the existence of *some* equilibrium supporting the given asset price process. In a continuous-time setting, the notion of no arbitrage unfortunately becomes very delicate (see Harrison and Kreps [14], Delbaen and Schachermayer [9, 10]) so one might expect the same to be true for the existence of a supporting equilibrium. However, following earlier work in this direction (Harrison and Kreps [14], Kreps [26], Ross [35]) in [17] a surprisingly simple answer is given: an equilibrium supporting a given price process exists if and only if the price process satisfies the No Free Lunch with Vanishing Risk (NFLVR) property of Delbaen and Schachermayer, as well as Merton's No Dominance (ND) condition [30], see also Jarrow, Protter, Shimbo [18, 19]. A key implication of this characterization is that informational efficiency can be rejected without committing to any particular equilibrium model by merely proving the existence of arbitrage opportunities and/or dominated strategies. At least for rejecting informational efficiency, this circumvents the joint model hypothesis problem.

For accepting informational efficiency, however, exhausting all possible arbitrage opportunities and/or dominated strategies is not practical. A second characterization of informational efficiency also proven in [17] can be used in this regard. It is shown that a supporting equilibrium exists if and only if an equivalent probability measure exists such that the price process, normalized by the money market account's value, is a martingale. This is called an *equivalent martingale measure*. With this characterization, informational efficiency can be accepted using a *conditional joint model hypothesis*. Indeed, one first assumes a particular stochastic process for the asset's price evolution. Then, this evolution is empirically validated/rejected. If this evolution is validated, then the stochastic processes' parameters are checked for consistency with the existence of an equivalent martingale measure. If consistent, then informational efficiency is accepted. In contrast to the joint model hypothesis, this approach has the advantage that the assumed evolution can be validated/rejected independently of informational efficiency.

A crucial assumption in the above formulation is that markets are frictionless, i.e. there are no transaction costs and trading is unrestricted. When this assumption provides a reasonable approximation, the above methodology is appropriate for testing informational efficiency. Yet, not for all markets and not for a given market at all times is trading unrestricted. Indeed, in most emerging markets short selling is impossible or not allowed (see Bris, Goetzmann, and Zhu [5]), and during the recent financial crisis regulators in well developed markets prohibited short sales to halt declining prices (see Beber and Pagano [3], Boehmer, Jones and Zhang [4]). When trading is constrained, using the above methodology to test for informational efficiency may lead to false rejections. The rejection being due to

the existence of trading constraints and not market inefficiency.

The purpose of the present paper is to extend the model in [17] to include trading constraints, more specifically short sale constraints. We now define a *constrained informationally efficient* market as one where there exists an equilibrium with agents facing short sale constraints that supports the given price process. This seemingly modest alteration brings new insights and subtleties in the characterization of an informationally efficient market.

Our first main result, Theorem 3.1, shows that constrained informational efficiency implies the existence of a local martingale deflator, turning asset prices into local martingales. This may come as a surprise since, due to the short sale constraint, only supermartingale deflators should be guaranteed. The key insight here is that market clearing always results in some agent being locally unconstrained, and this suffices to yield a local martingale deflator. It is well-known that the existence of a local martingale deflator does not rule out unconstrained arbitrage. Nonetheless, the above intuition regarding the role of market clearing suggests that unconstrained arbitrage strategies are also impossible in equilibrium—indeed, if such a strategy were to exist, an investor with strictly positive holdings could add “a little bit” of it, and thus do better without violating the short sale constraint. However, this reasoning fails. In Section 5.2 we give an example of an equilibrium with a single representative agent with logarithmic utility, optimally holding the full supply of the risky asset, but where arbitrage using unconstrained strategies is nonetheless possible (i.e., the condition NA fails.)

Less surprisingly, we find that constrained informational efficiency always implies a constrained version of NFLVR. However, the situation regarding ND is less clear. We show that a constrained version of ND holds whenever a representative agent equilibrium exists, but beyond that not much can be said. In fact, our strongest results are obtained in the setting of constrained representative agent equilibria. This situation is addressed by our second main result, Theorem 4.1, which shows that a given price process is supported by some such equilibrium if and only if, in addition to the above conditions, there is a local martingale deflator that turns the price process into a true martingale. Moreover, in a multi-agent complete market equilibrium, these conditions will be satisfied whenever all agents have strictly positive holdings in the risky asset, see Proposition 4.1. Finally, if a given equilibrium yields a price process that could, theoretically, be supported by some representative agent equilibrium, it is of interest to know more about this representative agent. In Theorem 4.2 we show that the representative agent can always be constructed by aggregating the individual agent utilities using the well-known procedure pioneered in Negishi [32].

With respect to a set of necessary and then sufficient conditions useful for testing constrained informational efficiency, our insights can be summarized as follows. A constrained informationally efficient market implies both No Unbounded Profit with Bounded Risk (NUPBR) and constrained NFLVR. If one can discover trading strategies violating either condition, then constrained informational efficiency can be rejected without the joint model hypothesis problem. In the converse direction, if there exists an equivalent supermartingale measure and a local martingale deflator that turns the price process into a martingale, then the market is constrained informationally efficient. In principle, this sufficient condition can be used, in conjunction with a validated price process, to accept an informationally efficient

market. Testing for constrained informational efficiency using these new insights awaits subsequent research.

An outline for this paper is as follows. Section 2 describes the pricing model and reviews the relevant no-arbitrage type conditions and their characterizations. Section 3 introduces the agents and the notion of equilibrium and constrained informational efficiency. Necessary conditions for efficiency are discussed. Section 4 deals with representative agent equilibria, and a characterization theorem is given. Aggregation of individual utilities is considered. Section 5 contains examples illustrating to what extent our results are sharp. Section 6 concludes and lists some open questions suggested by our analysis.

## 2 The Model

We consider a continuous time and continuous trading economy on a finite horizon. There are a finite number of agents in the economy, trading in a competitive market with no transaction costs. The market is constrained in that there are no short sales allowed for risky assets. Otherwise, it is assumed to be a frictionless market. We are given a complete filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where the filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ , defined on a bounded time interval  $[0, T]$ , satisfies the usual hypotheses. Here  $\mathbb{P}$  is the statistical probability measure. The traders in the economy have the information set  $\mathbb{F}$  and beliefs  $\mathbb{P}_k$  assumed to be equivalent to  $\mathbb{P}$ . It is this information set that is relevant for the subsequent definition of constrained informational efficiency.<sup>1</sup>

The financial market is assumed to consist of one risk-free asset and one risky asset. Prices are given in units of the risk-free asset, so that, in particular, the risk-free asset has a constant price equal to unity. The price at time  $t$  of the risky asset is denoted by  $S_t$ , and it is assumed that  $S = (S_t)_{0 \leq t \leq T}$  is a strictly positive continuous semimartingale with respect to the filtration  $\mathbb{F}$ .

A (self-financing) *trading strategy*  $H$  is an  $S$ -integrable predictable process representing the number of shares of the risky asset held at each point in time. It is called *a-admissible* if  $(H \cdot S)_t \geq -a$  for all  $t \in [0, T]$ , and *admissible* if it is *a-admissible* for some  $a \geq 0$ . It is called *constrained* if  $H_t \geq 0$  for all  $t \in [0, T]$ , i.e. if it satisfies a basic short sale restriction. We define

$$\begin{aligned} \mathcal{A} &= \{\text{all admissible strategies } H\}, \\ \mathcal{A}_C &= \{\text{all constrained admissible strategies } H\}. \end{aligned}$$

Note the slight abuse of terminology: in order to specify the trading strategy one also needs the number  $H_t^0$  of shares held in the risk-free asset at each time  $t$ . However, since only self-financing strategies are considered, the wealth process  $X_t = H_t^0 + H_t S_t$  satisfies

$$X_t = x + (H \cdot S)_t,$$

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<sup>1</sup>This structure can be extended to allow differential information across traders analogous to that discussed in [17], p. 11; see also [13] for related results. For simplicity of presentation, this extension is not discussed further herein.

where  $x = W_0$  is the initial capital. This relation determines  $H^0$ . Note that we do not impose short sale restrictions on the risk-free asset (see Cuoco [7] for a setting where such constraints are covered.)

## 2.1 No Arbitrage

Our primary interest is how short sale constraints affect no-arbitrage type restrictions in equilibrium. We now review the various notions of no arbitrage.

**Definition 2.1.** *We say that*

- NA ( $\text{NA}_C$ ) holds if  $(H \cdot S)_T \geq 0$  implies  $(H \cdot S)_T = 0$  for any  $H \in \mathcal{A}$  ( $H \in \mathcal{A}_C$ );
- ND ( $\text{ND}_C$ ) holds if  $(H \cdot S)_T \geq S_T - S_0$  implies  $(H \cdot S)_T = S_T - S_0$  for any  $H \in \mathcal{A}$  ( $H \in \mathcal{A}_C$ );
- NUPBR ( $\text{NUPBR}_C$ ) holds if the set  $\mathcal{K}$  ( $\mathcal{K}_C$ ) is bounded in  $\mathbb{L}^0$ , where

$$\begin{aligned}\mathcal{K} &= \{1 + (H \cdot S)_T : H \text{ is 1-admissible}\}, \\ \mathcal{K}_C &= \{1 + (H \cdot S)_T : H \text{ is constrained 1-admissible}\};\end{aligned}$$

- NFLVR ( $\text{NFLVR}_C$ ) holds if both NA and NUPBR ( $\text{NA}_C$  and  $\text{NUPBR}_C$ ) hold.

Here NA stands for No Arbitrage, ND for No Dominance, NUPBR for No Unbounded Profit with Bounded Risk, and NFLVR for No Free Lunch with Vanishing Risk. While NA is an old concept in finance, NFLVR was introduced in Delbaen and Schachermayer[9] and then in a more general setting in [10]. The condition NUPBR was studied in a general setting in Karatzas and Kardaras [21]. The ND condition is perhaps less well-known, but was in fact already introduced in Merton [30]. Its intuitive economic meaning is that it is impossible to find a trading strategy which dominates buying and holding the risky asset itself. We also note that by a well-known characterization result, see for instance [21, Proposition 4.2], NFLVR ( $\text{NFLVR}_C$ ) as defined above is equivalent to the condition that there is no sequence  $H^n \in \mathcal{A}$  ( $H^n \in \mathcal{A}_C$ ) such that  $f_n = (H^n \cdot S)_T$  satisfies  $\lim_n \|\max(-f_n, 0)\|_{\mathbb{L}^\infty} = 0$  and  $\lim_n f_n = f$  almost surely for some  $f \geq 0$  with  $\mathbb{P}(f > 0) > 0$ .

The following notion plays an important role in the analysis of the ND condition, among other things.

**Definition 2.2.** *A strategy  $H \in \mathcal{A}$  ( $H \in \mathcal{A}_C$ ) is called maximal (C-maximal) if  $(K \cdot S)_T \geq (H \cdot S)_T$  implies  $(K \cdot S)_T = (H \cdot S)_T$  for any  $K \in \mathcal{A}$  ( $K \in \mathcal{A}_C$ ).*

We can now phrase the ND condition as saying that holding a constant number of shares in the risky asset is a maximal strategy. Similarly, the NA condition means that investing all one's wealth in the risk-free asset is a maximal strategy.

## 2.2 Dual Characterizations

The various meanings of no arbitrage appearing in Definition 2.1 can be characterized in terms of equivalent probability measures, or, more generally, deflator processes. In order to state these results we introduce the following sets.

$$\begin{aligned}
\mathcal{M} &= \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q}\text{-martingale}\} \\
\mathcal{M}_{\text{loc}} &= \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a local } \mathbb{Q}\text{-martingale}\} \\
\mathcal{M}_{\text{sup}} &= \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q}\text{-supermartingale}\} \\
\mathcal{D}_{\text{loc}} &= \{Y : Y \text{ is càdlàg adapted, } Y \geq 0, Y_0 = 1, \text{ and } Y(1 + H \cdot S) \text{ is} \\
&\quad \text{a local martingale for every 1-admissible } H\} \\
\mathcal{D}_{\text{sup}} &= \{Y : Y \text{ is càdlàg adapted, } Y \geq 0, Y_0 = 1, \text{ and } Y(1 + H \cdot S) \text{ is} \\
&\quad \text{a supermartingale for every constrained 1-admissible } H\}.
\end{aligned}$$

The elements of  $\mathcal{M}$  are called *equivalent martingale measures*. The elements of  $\mathcal{M}_{\text{loc}}$  and  $\mathcal{M}_{\text{sup}}$  are called *equivalent local martingale (supermartingale) measures*, respectively, whereas the elements of  $\mathcal{D}_{\text{loc}}$  ( $\mathcal{D}_{\text{sup}}$ ) are called *local martingale (supermartingale) deflators*. If we identify a measure  $\mathbb{Q} \sim \mathbb{P}$  with its Radon-Nikodym density process, we have the following inclusions

$$\mathcal{M} \subset \mathcal{M}_{\text{loc}} \subset \mathcal{M}_{\text{sup}} \subset \mathcal{D}_{\text{sup}} \quad \text{and} \quad \mathcal{M} \subset \mathcal{M}_{\text{loc}} \subset \mathcal{D}_{\text{loc}} \subset \mathcal{D}_{\text{sup}}.$$

The no arbitrage type conditions from Definition 2.1 have the following characterizations.

**Theorem 2.1.** *The following assertions hold.*

- (i) NUPBR ( $\text{NUPBR}_C$ ) holds if and only if  $\mathcal{D}_{\text{loc}} \neq \emptyset$  ( $\mathcal{D}_{\text{sup}} \neq \emptyset$ );
- (ii) NFLVR ( $\text{NFLVR}_C$ ) holds if and only if  $\mathcal{M}_{\text{loc}} \neq \emptyset$  ( $\mathcal{M}_{\text{sup}} \neq \emptyset$ );
- (iii) NFLVR and ND hold if and only if  $\mathcal{M} \neq \emptyset$ .

*Proof.* Part (i) was proved in [21], Part (ii) in [9] (see [21] for the constrained case), and Part (iii) in [17].  $\square$

## 3 Informational Efficiency

We are interested in economies populated by agents who trade in the risky and risk-free assets in order to maximize expected utility from terminal wealth. We assume that the risky asset is available in unit net supply, while the risk-free asset is available in zero net supply. The agents face short sale restrictions on the risky asset, and therefore solve optimization problems of the form

$$u(x) = \sup_{H \in \mathcal{A}_C} \mathbb{E}[U(x + (H \cdot S)_T)]. \quad (3.1)$$

Here  $x > 0$  is the initial wealth, and  $U : \Omega \times (0, \infty) \rightarrow \mathbf{R}$  is a *stochastic utility function*. By this we mean that  $U$  is  $\mathcal{F}_T \otimes \mathcal{B}(\mathbf{R})$ -measurable,  $\mathbb{E}[|U(x)|] < \infty$  for all  $x > 0$ , and almost surely,  $x \mapsto U(x)$  is continuously differentiable, strictly increasing, strictly concave, with  $\lim_{x \rightarrow \infty} U(x) = \infty$ . It also satisfies Inada conditions at zero and infinity:  $\lim_{x \downarrow 0} U'(x) = \infty$ ,  $\lim_{x \rightarrow \infty} U'(x) = 0$ . Its domain of definition is extended to all of  $\mathbf{R}$  by setting  $U(x) = -\infty$  for  $x \leq 0$ . We use the convention that  $\mathbb{E}[U(x + (H \cdot S)_T)] = -\infty$  whenever the negative part of  $U(x + (H \cdot S)_T)$  has infinite expectation, even if the positive part also has infinite expectation.

The maximization problem (3.1) has an associated family of dual problems,

$$v(y) = \sup_{Y \in \mathcal{D}_{\text{sup}}} \mathbb{E}[V(yY_T)], \quad (3.2)$$

where  $V(y) = \sup_{x > 0} (U(x) - xy)$  is the conjugate of  $U$ , and  $y > 0$ . There is a large literature where existence and uniqueness of solutions to (3.1)–(3.2) is established, and the optimizers characterized, under a variety of assumptions on  $U$  and  $S$ . Rather than committing to a specific set of hypotheses, we will simply assume that our utility functions are sufficiently well-behaved that the main conclusions of the duality theory of (3.1)–(3.2) are valid. The main reasons for doing this are to avoid imposing artificial conditions on our utility functions, and to help emphasize which properties are, in fact, important for the subsequent results.

**Definition 3.1.** *We say that a stochastic utility function  $U$  is viable if, whenever the conditions*

$$\mathcal{M}_{\text{sup}} \neq \emptyset, \quad u(x) \in \mathbf{R} \text{ for some } x > 0, \quad v(y) \in \mathbf{R} \text{ for some } y > 0 \quad (3.3)$$

*hold, we have*

- (i)  *$u$  and  $v$  are conjugate:  $v(y) = \sup_{x > 0} (u(x) - xy)$  and  $u(x) = \inf_{y > 0} (v(y) + xy)$ . Furthermore, they are continuously differentiable on  $(0, \infty)$ .*
- (ii) *For each  $x > 0$ , the primal problem (3.1) has an optimal solution  $\hat{H} \in \mathcal{A}_C$ . If  $H$  is another optimal solution, then  $(H \cdot S)_T = (\hat{H} \cdot S)_T$ .*
- (iii) *For each  $y > 0$ , the dual problem (3.2) has a unique optimal solution  $\hat{Y} \in \mathcal{D}_{\text{sup}}$ .*
- (iv) *If  $x$  and  $y$  are related via  $y = u'(x)$ , then the corresponding optimal solutions satisfy*

$$x + (\hat{H} \cdot S)_T = I(y\hat{Y}_T),$$

*where  $I = (U')^{-1}$ . Moreover,  $\hat{Y}(x + \hat{H} \cdot S)$  is a martingale.*

Conditions under which viability holds are available in the extensive literature on utility maximization. For example, Karatzas and Zitkovic [23] give conditions that are applicable to the current setting (and allow the condition on  $v$  in (3.3) to be dropped.) The seminal paper in the unconstrained case is Kramkov and Schachermayer [25], and a more general



characterization of viability is given in Mostovy [31]. The latter paper suggests that in general one should include the condition on  $v$  in (3.3).

Let us briefly remark on the possibility of having stochastic utility functions. This is convenient for several reasons. First, it allows us to cover our assumption of heterogeneous beliefs  $\mathbb{P}^* \sim \mathbb{P}$  (replace  $U(x)$  by  $ZU(x)$ , where  $Zd\mathbb{P} = d\mathbb{P}^*$ .) Second, the initial discounting of the prices is without loss of generality (replace  $U(x)$  by  $U(\exp(\int_0^T r_t dt)x)$ , where  $r$  is the interest rate.) Third, the assumption that utility depends on final wealth can be replaced by the assumption that utility arises from consuming, at time  $T$ , a good whose unit price is  $\psi$  (replace  $U(x)$  by  $U(x\psi)$ .) Of course, the above statements all come with the caveat that viability must be preserved under the respective modifications of the utilities.

We now consider our  $n$  agents, indexed by  $k = 1, \dots, n$ . Each agent is equipped with a viable stochastic utility function  $U_k$  and some initial wealth  $x_k > 0$ . In equilibrium, the agent will follow some trading strategy  $\hat{H}^k \in \mathcal{A}_C$ . We refer to  $(U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  as the *constrained agent characteristics*. We use the following standard notion of an economic equilibrium.

**Definition 3.2.** *A constrained equilibrium is a financial market  $S$  together with constrained agent characteristics  $(U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  such that*

- (i) *individual optimality holds: For each  $k$ ,  $\hat{H}^k$  is an optimal solution to (3.1) with  $U = U_k$  and  $x = x_k$ , whose optimal value is finite. It is also required that the dual value function is finite for some  $y$ ;*
- (ii) *markets clear:  $\hat{H}_t^1 + \dots + \hat{H}_t^n = 1$  for all  $t \in [0, T]$  and  $x_1 + \dots + x_n = S_0$ .*

The aggregate wealth in the economy at time zero is  $S_0$ , since the risky asset is in unit net supply with initial price  $S_0$ , and the risk-free asset is in zero net supply. This explains the form of the market clearing condition for the initial wealth levels.

Note also that the market clearing condition (ii) automatically implies that market clearing holds for the risk-free asset. Indeed, the holdings of investor  $k$  in the risk-free asset is  $\hat{H}_t^{0,k} = x_k + (\hat{H}^k \cdot S)_t - \hat{H}_t^k S_t$  by the self-financing property. Therefore,

$$\sum_k \hat{H}_t^{0,k} = \sum_k x_k + (1 \cdot S)_t - S_t = S_0 + S_t - S_0 - S_t = 0.$$

We can now introduce the notion of *constrained informational efficiency*. It generalizes the notion of informational efficiency discussed in [17] to the case where short sale restrictions are present.

**Definition 3.3.** *A financial market  $S$  is called constrained informationally efficient if it supported by some constrained equilibrium, that is, if there exist constrained agent characteristics  $(U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  which together with  $S$  form a constrained equilibrium.*

Our aim is now to generate necessary conditions for constrained informational efficiency. Sufficient conditions will be investigated in Section 4, where the stronger property of the existence of a representative agent plays a key role.



**Lemma 3.1.** *Assume that for some  $x > 0$  the maximization problem (3.1) has an optimal solution  $\hat{H}$  with finite optimal value. Then*

- (i)  $\hat{H}$  is  $C$ -maximal;
- (ii)  $\text{NFLVR}_C$  holds.

*Proof.* (i): Let  $K$  be a constrained admissible strategy with  $\Delta := (K \cdot S)_T - (\hat{H} \cdot S)_T \geq 0$ . Since  $\hat{H}$  gives finite utility and since  $U_k$  strictly increasing,  $K$  would yield a strict improvement if  $\mathbb{P}(\Delta > 0) > 0$ . Hence  $\Delta = 0$ , and  $\hat{H}$  is  $C$ -maximal.

(ii): We need to establish  $\text{NA}_C$  and  $\mathbb{L}^0$ -boundedness of the set  $\mathcal{K}_C$  in Definition 2.1. The argument is well-known. First, suppose  $K$  is a constrained admissible strategy with  $(K \cdot S)_T \geq 0$ . Then  $\hat{H} + K$  is also constrained admissible, and we have  $((\hat{H} + K) \cdot S)_T \geq (\hat{H} \cdot S)_T$ . But  $\hat{H}$  is  $C$ -maximal by Part (i), so this inequality must be an equality. Hence  $(K \cdot S)_T = 0$ , and we deduce  $\text{NA}_C$ . To prove that  $\mathcal{K}_C$  is bounded in  $\mathbb{L}^0$ , we adapt [21, Proposition 4.19] to the case with stochastic utility functions. Assume for contradiction that  $\mathcal{K}_C$  is not bounded in  $\mathbb{L}^0$ . Then we can find  $\varepsilon > 0$  and constrained 1-admissible strategies  $H^n$ ,  $n \in \mathbb{N}$ , such that  $\mathbb{P}(1 + (H^n \cdot S)_T > n) > \varepsilon$ . Thus,

$$u(\varepsilon + 1) \geq \mathbb{E}[U(\varepsilon + 1 + (H^n \cdot S)_T)] \geq -\mathbb{E}[|U(\varepsilon)|] + [U(\varepsilon + 1 + n)\mathbf{1}_{\{1 + (H^n \cdot S)_T > n\}}].$$

Since  $U(\varepsilon + 1 + n) \rightarrow \infty$  almost surely, we can find, for any  $m > 0$ , some large  $n$  such that  $\mathbb{P}(U(\varepsilon + 1 + n) > m) > 1 - \varepsilon/2$ . Then,

$$U(\varepsilon + 1 + n)\mathbf{1}_{\{1 + (H^n \cdot S)_T > n\}} \geq m\mathbf{1}_{\{1 + (H^n \cdot S)_T > n \text{ and } U(\varepsilon + 1 + n) > m\}} - |U(\varepsilon)|,$$

so that

$$\begin{aligned} u(\varepsilon + 1) &\geq -2\mathbb{E}[|U(\varepsilon)|] + m\mathbb{P}(1 + (H^n \cdot S)_T > n \text{ and } U(\varepsilon + 1 + n) > m) \\ &\geq -2\mathbb{E}[|U(\varepsilon)|] + m\frac{\varepsilon}{2}. \end{aligned}$$

It follows that  $u(\varepsilon + 1) = \infty$ , which by concavity of  $u$  implies  $u(x) = \infty$ . This is the desired contradiction.  $\square$

We are now ready to give our first main result, which states that in a constrained equilibrium there is in fact a local martingale deflator. *A priori* one would only expect a supermartingale deflator to exist. However, market clearing implies that at every point in time, there must be some investor who optimally holds a positive number of shares in the risky asset. Such an investor is locally unconstrained, and this forces the price to behave, in a qualitative way, as it would in an unconstrained economy—otherwise the investor would reduce his holdings, contradicting optimality.

**Theorem 3.1.** *Let  $(S; U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  be a constrained equilibrium. Then  $\mathcal{D}_{\text{loc}}$  is non-empty.*

It is a striking fact that the above heuristic motivation for Theorem 3.1 fails if we are interested in the NA property rather than the NUPBR property. Indeed, in Section 5.2 we provide an example where both NFLVR<sub>C</sub> and NUPBR hold, NA fails, and a strictly positive strategy  $\hat{H}$  (in fact, the constant strategy  $\hat{H} \equiv 1$ ) is optimal for (3.1). In other words, even though the optimal strategy is strictly positive, the short sale constraint is binding in the sense that an unconstrained agent would choose a very different strategy.

An immediate consequence of Theorem 3.1 and Lemma 3.1(ii) is the following.

**Corollary 1.** *A constrained informationally efficient market  $S$  necessarily satisfies both NFLVR<sub>C</sub> and NUPBR.*

The proof of Theorem 3.1 relies on the following lemma.

**Lemma 3.2.** *Let  $U$  be a viable stochastic utility function. Suppose (3.3) holds, and let  $\hat{Y}$  be the optimal solution to the dual problem (3.2) for some  $y > 0$ . Then  $\hat{Y}$  is a strictly positive local martingale.*

*Proof.* The Inada conditions imply  $I(0) = \infty$ , which forces  $\hat{Y}_T > 0$  due to (iv) in the definition of viability. Since  $\hat{Y}$  is a nonnegative supermartingale this yields  $\hat{Y} > 0$ , proving the first assertion. As a consequence (and since  $\hat{Y}_0 = 1$ ), there is a local martingale  $N$  and a nondecreasing predictable process  $B$  such that  $\hat{Y} = \mathcal{E}(-N - B)$ . We need to prove that  $B = 0$ . To this end, observe that since  $S > 0$  there is a continuous semimartingale  $M + A$ , with  $M$  a local martingale and  $A$  a predictable finite variation process, such that  $S = S_0 \mathcal{E}(M + A)$ . Furthermore, for any nonnegative predictable  $(M + A)$ -integrable process  $\theta$ ,  $X = \mathcal{E}(\theta \cdot (M + A))$  is the wealth process of the constrained 1-admissible strategy  $H = S^{-1} \theta X$ . Indeed, we have  $H \geq 0$ ,  $X > 0$ , and

$$X = 1 + X \cdot (\theta \cdot (M + A)) = 1 + (X \theta S^{-1}) \cdot (S \cdot (M + A)) = 1 + H \cdot S.$$

Since  $\hat{Y} \in \mathcal{D}_{\text{sup}}$ , therefore,  $\hat{Y}X$  is a supermartingale. Moreover, Yor's formula and the fact that  $[M, B] = [A, B] = [A, N] = 0$  due to the continuity of  $M$  and  $A$ , yield

$$\hat{Y}X = \mathcal{E}(-N - B) \mathcal{E}(\theta \cdot (M + A)) = \mathcal{E}(-N + \theta \cdot M + \theta \cdot (A - [N, M]) - B).$$

It follows that  $\theta \cdot (A - [N, M]) - B$  is a non-increasing process for any  $\theta \geq 0$ , and thus  $A - [N, M]$  must also be non-increasing. We claim that this implies that  $Y' = \mathcal{E}(-N)$  lies in  $\mathcal{D}_{\text{sup}}$ . To see why, note that for any constrained 1-admissible  $H$  we have, with  $X = 1 + H \cdot S$ ,

$$\begin{aligned} Y'X &= 1 + Y'_- \cdot X + X \cdot Y' + [Y', X] \\ &= 1 + (Y'_- HS) \cdot (M + A) + X \cdot Y' - (Y'_- HS) \cdot [N, M] \\ &= 1 + (Y'_- HS) \cdot M + X \cdot Y' + (Y'_- HS) \cdot (A - [N, M]). \end{aligned}$$

Only the last term is not a local martingale, and it is non-increasing since  $Y'_- HS \geq 0$  and  $A - [N, M]$  is non-increasing. So  $Y' \in \mathcal{D}_{\text{sup}}$  as claimed. But we also have

$$\hat{Y}_T = \mathcal{E}(-N - B)_T \leq \mathcal{E}(-N)_T = Y'_T,$$

with strict inequality on the set  $\{B_T > 0\}$ . If this set had positive probability,  $Y'$  would achieve a strictly smaller objective value than  $\hat{Y}$  in the dual problem (3.2), contradicting the optimality of  $\hat{Y}$ . Hence  $B_T = 0$ , and the lemma is proved.  $\square$

*Proof of Theorem 3.1.* Let  $\hat{Y}^k$  be the dual optimizer of the  $k$ :th investor's problem. By Lemma 3.2 it is a strictly positive local martingale, and condition (iv) in the definition of viability implies that  $\hat{Y}^k(x_k + \hat{H}^k \cdot S)$  is a martingale. In particular we may write  $\hat{Y}^k = \mathcal{E}(-N^k)$  for some local martingale  $N^k$ . Letting  $S = M + A$  be the canonical decomposition of  $S$ , integration by parts yields

$$\begin{aligned}\hat{Y}^k(x_k + \hat{H}^k \cdot S) &= x_k + (x_k + \hat{H}^k \cdot S) \cdot \hat{Y}_-^k + (\hat{Y}^k \hat{H}^k) \cdot (M + A) + \hat{H}^k \cdot [\hat{Y}^k, M] \\ &= x_k + (x_k + \hat{H}^k \cdot S) \cdot \hat{Y}_-^k + (\hat{Y}_-^k \hat{H}^k) \cdot M + (\hat{H}^k \hat{Y}_-^k) \cdot (A - [N^k, M]).\end{aligned}$$

Since the left side is a  $\mathbb{P}$ -martingale, and since  $\hat{Y}^k > 0$ , we deduce  $\hat{H}^k \cdot (A - [N^k, M]) = 0$ , and hence

$$\mathbf{1}_{\{\hat{H}^k > 0\}} \cdot (A - [N^k, M]) = 0. \quad (3.4)$$

We now define predictable sets  $D_k$  by

$$D_1 = \{\hat{H}^1 > 0\}, \quad D_k = \{\hat{H}^k > 0\} \setminus D_{k-1}, \quad k = 2, \dots, n.$$

Then clearly  $D_k \cap D_\ell = \emptyset$  whenever  $k \neq \ell$ . Moreover, market clearing implies that for every  $(t, \omega)$ , there is at least one  $k \in \{1, \dots, n\}$  such that  $\hat{H}_t^k(\omega) > 0$ . Therefore  $\cup_k D_k = \cup_k \{\hat{H}^k > 0\} = [0, T] \times \Omega$ , and hence

$$\mathbf{1}_{D_1} + \dots + \mathbf{1}_{D_n} = 1.$$

Finally, since  $D_k \subset \{\hat{H}^k > 0\}$ , it follows from (3.4) that  $\mathbf{1}_{D_k} \cdot (A - [N^k, M]) = 0$  for each  $k$ . Thus with

$$N = \mathbf{1}_{D_1} \cdot N^1 + \dots + \mathbf{1}_{D_n} \cdot N^n,$$

we get

$$A - [N, M] = \sum_{k=1}^n \mathbf{1}_{D_k} \cdot (A - [N^k, M]) = 0.$$

Setting  $Y = \mathcal{E}(-N)$  this yields  $YS = S \cdot Y + Y \cdot M$ , which is a local martingale. Thus  $Y \in \mathcal{D}_{\text{loc}}$ , and the theorem is proved.  $\square$

## 4 Representative Agent Equilibria

In this section we consider financial markets that are not only informationally efficient in the sense of Definition 3.3, but have the additional feature that a representative agent can be constructed. Such markets turn out to have a simple dual characterization in terms of supermartingale measures and local martingale deflators, see Theorem 4.1 below. We also consider the question of aggregation of a multi-agent equilibrium to a representative agent equilibrium. This issue has a long history in the literature, see for instance Magill[29], Constantinides [6], Karatzas, Lehoczky, and Shreve [22], Cuoco and He [8].

**Definition 4.1.** A constrained representative agent equilibrium  $(S; U)$  is a constrained equilibrium with one single agent. That is, the constrained agent characteristics is  $(U, 1, 1)$ .

By market clearing, a representative agent in any equilibrium (constrained or not) must use the constant strategy  $\hat{H} \equiv 1$ . This strategy automatically lies in  $\mathcal{A}_C$ , clearly satisfying the short sale constraint. However, there exist constrained representative agent equilibria, where the constant strategy is optimal in the class  $\mathcal{A}_C$ , but not in the class  $\mathcal{A}$  of unconstrained admissible strategies. An example is given in Section 5.2.

The following is the main result of this section.

**Theorem 4.1.** Consider a financial market  $S$ . The following conditions are equivalent:

- (i)  $(S, U)$  is a constrained representative agent equilibrium for some viable stochastic utility function  $U$ ,
- (ii)  $\mathcal{M}_{\text{sup}} \neq \emptyset$  and there exists a process  $Z \in \mathcal{D}_{\text{loc}}$  such that  $ZS$  is a martingale.

In either case,  $S$  satisfies NFLVR<sub>C</sub>, ND<sub>C</sub>, and NUPBR.

An immediate consequence of this theorem and the definition of constrained informational efficiency is the following.

**Corollary 2.** If  $\mathcal{M}_{\text{sup}} \neq \emptyset$  and there exists a process  $Z \in \mathcal{D}_{\text{loc}}$  such that  $ZS$  is a martingale, then the market is constrained informationally efficient.

The following lemma is used in the proof of Theorem 4.1. It essentially shows that viability is invariant under changes of beliefs. See Källblad [20, Theorem 1] for a related result.

**Lemma 4.1.** Let  $U^*$  be a viable stochastic utility function, and  $\mathbb{P}^* \sim \mathbb{P}$  an equivalent probability measure. Set  $Z_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}|_{\mathcal{F}_t}$ . Then the stochastic utility function  $U = Z_T U^*$  is again viable.

*Proof.* Since the class  $\mathcal{A}_C$  is invariant under equivalent changes of probability measure, we have

$$u(x) = \sup_{H \in \mathcal{A}_C} \mathbb{E}[U(x + (H \cdot S)_T)] = \sup_{H \in \mathcal{A}_C} \mathbb{E}^*[U^*(x + (H \cdot S)_T)], \quad (4.1)$$

where  $\mathbb{E}^*[\cdot]$  denotes expectation under  $\mathbb{P}^*$ . Furthermore, we have

$$V(y) = \sup_{x > 0} (U(x) - xy) = Z_T \sup_{x > 0} \left( U^*(x) - x \frac{y}{Z_T} \right) = Z_T V^* \left( \frac{y}{Z_T} \right),$$

where  $V^*$  is the conjugate of  $U^*$ . Therefore,

$$v(y) = \inf_{Y \in \mathcal{D}_{\text{sup}}} \mathbb{E} \left[ Z_T V^* \left( \frac{y Y_T}{Z_T} \right) \right] = \inf_{Y^* \in \mathcal{D}_{\text{sup}}^*} \mathbb{E}^* [V^*(y Y_T^*)], \quad (4.2)$$

where  $\mathcal{D}_{\text{sup}}^* = \{Y/Z : Y \in \mathcal{D}_{\text{sup}}\}$ . Note that this is the set of all càdlàg adapted  $Y^*$  with  $Y^* \geq 0$  and  $Y_0^* = 1$  such that  $Y^*(1 + H \cdot S)$  is a  $\mathbb{P}^*$ -supermartingale for every constrained 1-admissible  $H$ . Suppose now (3.3) holds, and note that this condition is invariant under equivalent changes of measure. In view of (4.1) and (4.2), the viability of  $U^*$  implies that  $u$  and  $v$  are conjugate and continuously differentiable. Moreover, the primal problem has an optimal solution  $\hat{H} \in \mathcal{A}_C$ , and the dual problem an optimal solution  $\hat{Y}^* \in \mathcal{D}_{\text{sup}}^*$ . Both are unique in the appropriate sense, see Definition 3.1(ii)–(iii). The process  $\hat{Y} = Z\hat{Y}^*$  then lies in  $\mathcal{D}_{\text{sup}}$  and is optimal for the dual problem under  $\mathbb{P}$ . At this point we have established properties (i)–(iii) in the definition of viability. Property (iv) follows from the equality

$$x + (\hat{H} \cdot S)_T = I^*(yY_T^*) = I(yZ_TY_T^*) = I(yY_T)$$

and the fact that  $\hat{Y}(x + \hat{H} \cdot S)$  is a  $\mathbb{P}$ -martingale if and only if  $\hat{Y}^*(x + \hat{H} \cdot S)$  is a  $\mathbb{P}^*$ -martingale. The lemma is proved.  $\square$

*Proof of Theorem 4.1.* (i)  $\implies$  (ii): Lemma 3.1 implies that NFLVR $_C$  holds, so that  $\mathcal{M}_{\text{sup}} \neq \emptyset$  by Theorem 2.1. Market clearing together with viability of  $U$  show that the dual optimizer  $Z = \hat{Y}$  has the property that  $ZS$  is a martingale. Moreover,  $Z$  is a local martingale by Lemma 3.2. This implies that  $Y \in \mathcal{D}_{\text{loc}}$ . Indeed, let  $H \in \mathcal{A}_C$  be 1-admissible. Integration by parts applied twice (and associativity of the stochastic integral) yields

$$\begin{aligned} Z(1 + H \cdot S) &= 1 + (1 + H \cdot S) \cdot Z + Z_- \cdot (H \cdot S) + H \cdot [Z, S] \\ &= 1 + (1 + H \cdot S - HS) \cdot Z + H \cdot (ZS). \end{aligned}$$

Thus the left side is a stochastic integral with respect to the (two-dimensional) local martingale  $(Z, ZS)$ . Since it is nonnegative it is again a local martingale by the Ansel-Stricker theorem.

(ii)  $\implies$  (i): The proof is inspired by [17, Theorem 3.2]. As a candidate utility function for the representative agent we take

$$U(x) = \frac{Z_T S_T^\gamma x^{1-\gamma}}{1-\gamma}$$

for some  $\gamma \in (0, 1)$ . Let us show that  $\hat{H} \equiv 1$  is optimal. Clearly  $\hat{H} \in \mathcal{A}_C$ . Next, the corresponding utility is  $\mathbb{E}[U(S_T)] = \mathbb{E}[Z_T S_T]/(1-\gamma) = 1/(1-\gamma)$ , in particular it is finite. Now let  $H \in \mathcal{A}_C$  be arbitrary. We may assume that the final wealth  $1 + (H \cdot S)_T$  is positive almost surely, otherwise the utility would be  $-\infty$ . This implies that the wealth process is in fact 1-admissible. Indeed,  $H \cdot S$  is a supermartingale under any  $\mathbb{Q} \in \mathcal{M}_{\text{sup}}$ , which exists by assumption, so

$$1 + (H \cdot S)_t \geq \mathbb{E}^\mathbb{Q}[1 + (H \cdot S)_T \mid \mathcal{F}_t] > 0. \quad (4.3)$$

Here we used [34, Proposition 3.5], the “Ansel-Stricker theorem for supermartingales”. It is now easy to see that  $\hat{H}$  is optimal. Indeed, concavity of  $U$ , the equality  $U'(S_T) = Z_T$ , and the properties of  $Z$  yield

$$\mathbb{E}[U(1 + (H \cdot S)_T)] \leq \mathbb{E}[U(S_T)] + \mathbb{E}[U'(S_T)(1 + (H \cdot S)_T)] - \mathbb{E}[U'(S_T)S_T]$$

$$\begin{aligned}
&= \mathbb{E}[U(S_T)] + \mathbb{E}[Z_T(1 + (H \cdot S)_T)] - \mathbb{E}[Z_T S_T] \\
&\leq \mathbb{E}[U(S_T)].
\end{aligned}$$

We need to prove that  $U$  is viable. To this end, write  $U(x) = \frac{d\mathbb{P}^*}{d\mathbb{P}} U^*(x)$ , where

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{Z_T S_T^\gamma}{\mathbb{E}[Z_T S_T^\gamma]}, \quad U^*(x) = \mathbb{E}[Z_T S_T^\gamma] \frac{x^{1-\gamma}}{1-\gamma}.$$

Note that  $\mathbb{E}[Z_T S_T^\gamma] \leq \mathbb{E}[Z_T(1 + S_T)] < \infty$ . The utility function  $U^*$  is viable by [23, Theorem 3.10], so Lemma 4.1 implies that  $U$  is also viable.

It now only remains to prove that one (and hence both) of (i) and (ii) implies NFLVR<sub>C</sub>, ND<sub>C</sub>, and NUPBR<sub>C</sub>. In view of what we have already done, only ND<sub>C</sub> needs to be verified. But this follows directly from Lemma 3.1(i).  $\square$

In the complete market setting one expects that a representative agent equilibrium can be found. The following result confirms this intuition.

**Proposition 4.1.** *Let  $(S; U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  be a constrained equilibrium. If  $\mathcal{D}_{\text{loc}}$  is a singleton, and  $\hat{H}^k > 0$  for all  $k$ , then the two equivalent conditions of Theorem 4.1 hold.*

*Proof.* Let  $\hat{Y}^k$  be the dual optimizers for the agents' optimization problems. By Lemma 3.2 there exist local martingales  $N^k$  such that  $\hat{Y}^k = \mathcal{E}(-N^k)$ . Let  $S = M + A$  be the canonical decomposition of  $S$ . Integration by parts yields

$$\hat{Y}^k(x_k + \hat{H}^k \cdot S) = (x_k + \hat{H}^k \cdot S) \cdot \hat{Y}^k + \hat{Y}^k \hat{H}^k \cdot (M + A - [N^k, M]).$$

Since  $\hat{Y}^k(x_k + \hat{H}^k \cdot S)$  is a martingale, and since  $\hat{Y}^k \hat{H}^k > 0$ , we get  $A = [N^k, M]$ . Another application of the integration by parts formula shows that  $\hat{Y}^k S$  is a local martingale, so we deduce  $\hat{Y}^k \in \mathcal{D}_{\text{loc}} = \{Z\}$  and hence  $\hat{Y}^k = Z$  for all  $k$ . Market clearing gives

$$ZS = Z \sum_{k=1}^n (x_k + \hat{H}^k \cdot S) = \sum_{k=1}^n \hat{Y}^k(x_k + \hat{H}^k \cdot S),$$

which is a (true) martingale. Since also  $\mathcal{M}_{\text{sup}} \neq \emptyset$  by Lemma 3.1, condition (ii) of Theorem 4.1 is satisfied.  $\square$

Theorem 4.1 characterizes those price processes  $S$  for which a supporting representative agent equilibrium exists. However, the choice of representative agent utility  $U$  is far from unique in general, and it is natural to ask if some choices are more natural (or useful) than others. In particular, if the price process  $S$  is known to come from a constrained equilibrium  $(S; U_k, x_k, \hat{H}^k : k = 1, \dots, n)$ , it is of interest to know whether  $U_1, \dots, U_n$  can be *aggregated* to form a representative utility  $U$ . This was first done in Negishi [32] in the complete market case, see also Magill[29], Constantinides [6], Karatzas, Lehoczky, and Shreve [22], and in Cuoco and He [8] for incomplete markets. Extensions involving investment constraints have



been considered in Basak and Cuoco [2] and Hugonnier [16]. Specifically, one considers functions

$$U(x; \lambda) = \max_{c_1 + \dots + c_n = x} \sum_{k=1}^n \lambda_k U_k(c_k), \quad (4.4)$$

for weight vectors  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}_{++}^n$ , possibly stochastic, and asks for a weight vector such that the constant strategy is optimal for  $U(\cdot; \lambda)$  given the prevailing price process  $S$ . In this case we call  $U(\cdot; \lambda)$  an *aggregate utility function*.

**Theorem 4.2.** *Let  $(S; U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  be a constrained equilibrium supporting a representative agent—that is,  $S$  satisfies one of the equivalent conditions in Theorem 4.1. Then there are stochastic weights  $\lambda = (\lambda_1, \dots, \lambda_n)$  such that  $U(\cdot; \lambda)$  is an aggregate utility function.*

*Proof.* Fix any weights  $\lambda = (\lambda_1, \dots, \lambda_n)$  and any  $x > 0$ . The Lagrangian corresponding to the optimization problem in (4.4) is

$$L(c_1, \dots, c_n; \mu) = \sum_{k=1}^n \lambda_k U_k(c_k) + \mu \left( x - \sum_{k=1}^n c_k \right),$$

where  $\mu \in \mathbf{R}$  is the Lagrange multiplier. By strict concavity, the maximizer  $(c_1^*, \dots, c_n^*)$  and associated Lagrange multiplier  $\mu^*$  are characterized by the first order conditions, namely

$$x = c_1^* + \dots + c_n^*, \quad \lambda_k U'_k(c_k^*) - \mu^* = 0 \quad (k = 1, \dots, n). \quad (4.5)$$

Now, let  $Z \in \mathcal{D}_{\text{loc}}$  be such that  $ZS$  is a martingale—such  $Z$  exists by hypothesis. Moreover, let  $\hat{X}_T^k = x_k + (\hat{H}^k \cdot S)_T$  be the final wealth of agent  $k$ , and set

$$\lambda_k = \frac{Z_T}{U'_k(\hat{X}_T^k)}, \quad k = 1, \dots, n.$$

Since also  $\hat{X}_T^1 + \dots + \hat{X}_T^n = S_T$  holds by market clearing, (4.5) is satisfied with  $c_k^* = \hat{X}_T^k$  and  $\mu^* = Z_T$ . Hence

$$U(S_T; \lambda) = \lambda_1 U_1(\hat{X}_T^1) + \dots + \lambda_n U_n(\hat{X}_T^n). \quad (4.6)$$

Consider an arbitrary  $H \in \mathcal{A}_C$  with strictly positive final value  $X_T = 1 + (H \cdot S)_T$ . Then, for some  $X_T^1, \dots, X_T^n$  with  $X_T^1 + \dots + X_T^n = X_T$ , we have

$$\begin{aligned} U(X_T; \lambda) &= \lambda_1 U_1(X_T^1) + \dots + \lambda_n U_n(X_T^n) \\ &\leq \sum_{k=1}^n \left[ \lambda_k U_k(\hat{X}_T^k) + \lambda_k U'_k(\hat{X}_T^k)(X_T^k - \hat{X}_T^k) \right] \\ &= U(S_T; \lambda) + \sum_{k=1}^n \lambda_k U'_k(\hat{X}_T^k)(X_T^k - \hat{X}_T^k) \end{aligned}$$

$$= U(S_T; \lambda) + Z_T(X_T - S_T),$$

where we used the concavity of  $U_k$ , the identity (4.6), as well the fact that  $Z_T = \lambda_k U'_k(\widehat{X}_T^k)$  for all  $k$ . Using that  $X$  is strictly positive by NFLVR<sub>C</sub> (see (4.3) in the proof of Theorem 4.1), so that  $ZX$  is a positive local martingale and thus a supermartingale, we get

$$\mathbb{E}[U(X_T; \lambda)] \leq \mathbb{E}[U(S_T; \lambda)] + \mathbb{E}[Z_T(X_T - S_T)] \leq \mathbb{E}[U(S_T; \lambda)].$$

Thus the constant strategy is optimal for  $U(\cdot; \lambda)$ , which is what we had to prove.  $\square$

## 5 Examples

In this section we provide examples illustrating to what extent our previous results are sharp. The examples are based on Proposition 5.1 below, which is also interesting in its own right. We work on the canonical path space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where the coordinate process  $W$  is Brownian motion under  $\mathbb{P}$ , and  $\mathbb{F}$  is its natural filtration. Some care is needed in the description of  $\Omega$ : it is the set of all functions  $\omega : \mathbf{R}_+ \rightarrow \mathbf{R} \cup \{\infty\}$  that are continuous on  $[0, \zeta(\omega))$ , where  $\zeta(\omega) = \inf\{t \geq 0 : \omega(t) = \infty\}$ , and satisfy  $\omega(t) = \infty$  for all  $t \geq \zeta(\omega)$ . Moreover, the filtration  $\mathbb{F}$  is not augmented with the  $\mathbb{P}$ -nullsets, and this does not affect the stochastic calculus used below.

By means of a construction due to Föllmer [12], in turn inspired by Doob's  $h$ -transform, any strictly positive local martingale  $Z$  with  $Z_0 = 1$  can be viewed as the density process of some probability measure, possibly not equivalent with respect to  $\mathbb{P}$ . Specifically, define its explosion time by

$$\sigma = \lim_n \sigma_n, \quad \sigma_n = \inf\{t \geq 0 : Z_t \geq n\},$$

which of course satisfies  $\mathbb{P}(\sigma < \infty) = 0$ . Then there exists a probability measure  $\mathbb{P}^*$  on  $\mathcal{F}$  with  $\mathbb{P}|_{\mathcal{F}_t} \ll \mathbb{P}^*|_{\mathcal{F}_t}$  for each  $t \geq 0$ , such that

$$\left. \frac{d\mathbb{P}}{d\mathbb{P}^*} \right|_{\mathcal{F}_t} = \frac{1}{Z_t} \mathbf{1}_{\{t < \sigma\}},$$

and the equality  $\mathbb{E}[Z_t] = \mathbb{P}^*(\sigma > t)$  holds for each  $t \geq 0$ . In particular,  $Z$  is a true martingale under  $\mathbb{P}$  if and only if  $\mathbb{P}^*(\sigma = \infty) = 1$ . We refer to  $\mathbb{P}^*$  as the *Föllmer measure* corresponding to  $Z$ . The Föllmer measure is uniquely determined on  $\mathcal{F}_\sigma$ , but may have several different extensions to  $\mathcal{F}$ . This possible non-uniqueness plays a key role in Larsson[27] and is also discussed in Perkowski and Ruf [33]. For us the particular choice of extension is not important.

Girsanov's theorem also extends to this setting. We will only need the following simple version. If the local martingale  $Z$  is of the form  $Z = \mathcal{E}(\theta \cdot W)$  for some predictable  $W$ -integrable process  $\theta$ , there is a  $\mathbb{P}^*$ -Brownian motion  $W^*$ , possibly defined on an extension of the original probability space, such that

$$W_t^* = W_t - \int_0^t \theta_s ds, \quad t < \sigma. \quad (5.1)$$

Indeed, Girsanov's theorem shows that  $W_{t \wedge \sigma_n}^*$  is stopped Brownian motion under the equivalent measure  $\mathbb{P}^n$  with density process  $Z_{t \wedge \sigma_n}$  (this coincides with  $\mathbb{P}^*$  on  $\mathcal{F}_{\sigma_n}$ ), so  $W_t^*$  is well-defined on  $\llbracket 0, \sigma \rrbracket$ . Using that  $\langle W^*, W^* \rangle_t = t$ , one shows that  $\lim_{t \uparrow \sigma} W_t^*$  exists  $\mathbb{P}^*$ -a.s. on  $\{\sigma < \infty\}$ . Thus, letting  $W^{**}$  be an independent Brownian motion (this is where we may have to extend the probability space), the process  $W^* \mathbf{1}_{\llbracket 0, \sigma \rrbracket} + (W_t^{**} - W_\sigma^{**} + W_\sigma^*) \mathbf{1}_{\llbracket \sigma, \infty \rrbracket}$  is Brownian motion under  $\mathbb{P}^*$  and satisfies (5.1). For further details regarding the construction of the Föllmer measure and Girsanov's theorem, see Yoeurp[37].

The following proposition is the basis for our examples below. It is inspired by examples by Kazamaki, see [24, Chapter 1.4]. To state the result, define

$$p(T) = \mathbb{P}\left(\inf_{0 \leq t \leq T} W_t \leq -1\right),$$

the probability that standard Brownian motion hits  $-1$  before time  $T$ .

**Proposition 5.1.** *Fix  $T > 0$  and a constant  $\beta > 1$ . Let  $X$  be the unique strong solution to*

$$dX_t = -X_t^2 dW_t, \quad X_0 = 1,$$

*and define a process  $L$  and stopping time  $\tau$  by*

$$L = \frac{\mathcal{E}(-\beta X \cdot W)}{\mathcal{E}(-X \cdot W)} \quad \text{and} \quad \tau = \inf \{t \geq 0 : L_t \geq 1 + p(T)^{-1}\}.$$

*Then the local martingales  $Z^{(1)} = \mathcal{E}(-X \mathbf{1}_{\llbracket 0, \tau \rrbracket} \cdot W)$  and  $Z^{(\beta)} = \mathcal{E}(-\beta X \mathbf{1}_{\llbracket 0, \tau \rrbracket} \cdot W)$  satisfy  $\mathbb{E}[Z_T^{(1)}] < 1$  and  $\mathbb{E}[Z_T^{(\beta)}] = 1$ .*

The idea of the construction is best understood by passing to the Föllmer measures  $\mathbb{P}^{(1)}$  and  $\mathbb{P}^{(\beta)}$  corresponding to  $Z^{(1)}$  and  $Z^{(\beta)}$ , respectively. It is not hard to see that if  $\tau$  were always infinite, both these density processes would explode under their respective Föllmer measure, and would thus both be strict local martingales under  $\mathbb{P}$ . Of course  $\tau$  is not always infinite—in fact, it is designed to occur early enough to  $\mathbb{P}^{(\beta)}$ -almost surely prevent  $Z^{(\beta)}$  from exploding, while at the same time occurring late enough that  $Z^{(1)}$  does explode with positive  $\mathbb{P}^{(1)}$ -probability. The device for achieving this is the “likelihood ratio” process  $L$  (note that we have  $L_t = Z_t^{(\beta)} / Z_t^{(1)}$  for  $t < \tau$ .) It turns out that  $L$  is a supermartingale under  $\mathbb{P}^{(1)}$ , and hence fails to trigger  $\tau$  with positive probability, but is an exploding submartingale under  $\mathbb{P}^{(\beta)}$  (or, more precisely, would have exploded without the intervention of the stopping time  $\tau$ .) We now turn to the details.

*Proof.* We first prove  $\mathbb{E}[Z_T^{(1)}] < 1$ . Let  $\mathbb{P}^{(1)}$  be the Föllmer measure associated with  $Z^{(1)}$ , and let  $W^{(1)}$  be a  $\mathbb{P}^{(1)}$ -Brownian motion such that

$$W_t^{(1)} = W_t + \int_0^t X_s \mathbf{1}_{\{s < \tau\}} ds, \quad t < \sigma^{(1)},$$

where  $\sigma^{(1)}$  is the explosion time of  $Z^{(1)}$ . A calculation yields, for all  $t < \sigma^{(1)}$ , the equalities  $X_{t \wedge \tau} = \mathcal{E}(-X \cdot W)_{t \wedge \tau} = Z_t^{(1)}$  and  $\frac{1}{X_{t \wedge \tau}} = 1 + W_{t \wedge \tau} + \int_0^{t \wedge \tau} X_s ds = 1 + W_{t \wedge \tau}^{(1)}$ . This implies

$$\begin{aligned} 1 - \mathbb{E}[Z_T^{(1)}] &= \mathbb{P}^{(1)}(\sigma^{(1)} \leq T) \\ &\geq \mathbb{P}^{(1)}(1 + W^{(1)} \text{ hits zero before } T) - \mathbb{P}^{(1)}(\tau \leq T) \\ &= p(T) - \mathbb{P}^{(1)}(\tau \leq T). \end{aligned}$$

It suffices to show that the right side is strictly positive. To this end, we compute, for  $t < \sigma^{(1)}$ ,

$$\begin{aligned} L_t &= \exp \left( -(\beta - 1) \int_0^t X_s dW_s - \frac{1}{2} \int_0^t (\beta^2 - 1) X_s^2 ds \right) \\ &= \exp \left( -(\beta - 1) \int_0^t X_s dW_s^{(1)} - \frac{1}{2} \int_0^t [(\beta^2 - 1) X_s^2 - 2(\beta - 1) X_s^2 \mathbf{1}_{\{s < \tau\}}] ds \right) \\ &= \exp \left( -(\beta - 1) \int_0^t X_s dW_s^{(1)} - \frac{1}{2} \int_0^t (\beta - 1)^2 X_s^2 ds - \int_{t \wedge \tau}^t (\beta - 1) X_s^2 ds \right) \\ &= \mathcal{E}(-(\beta - 1)X \cdot W^{(1)})_t \exp \left( -(\beta - 1) \int_{t \wedge \tau}^t X_s^2 ds \right). \end{aligned}$$

Thus  $L$  is a supermartingale under  $\mathbb{P}^{(1)}$ , which implies

$$\mathbb{P}^{(1)}(\tau < \infty) = \mathbb{P}^{(1)}\left(\sup_{t \geq 0} L_t \geq 1 + p(T)^{-1}\right) \leq (1 + p(T)^{-1})^{-1} < p(T),$$

by Doob's inequality. This proves  $\mathbb{E}[Z_T^{(1)}] < 1$ .

We now establish  $\mathbb{E}[Z_T^{(\beta)}] = 1$ . Similarly as before, let  $\mathbb{P}^{(\beta)}$  be the Föllmer measure associated with  $Z^{(\beta)}$ , and let  $W^{(\beta)}$  be a  $\mathbb{P}^{(\beta)}$ -Brownian motion such that

$$W_t^{(\beta)} = W_t + \int_0^t \beta X_s \mathbf{1}_{\{s < \tau\}} ds, \quad t < \sigma^{(\beta)}$$

holds, where  $\sigma^{(\beta)}$  is the explosion time of  $Z^{(\beta)}$ . We then have

$$\frac{1}{Z_t^{(\beta)}} = \mathcal{E}(\beta X \cdot W^{(\beta)})_{t \wedge \tau}, \quad t < \sigma^{(\beta)},$$

which implies  $\mathbb{P}^{(\beta)}(\tau < \sigma^{(\beta)} < \infty) = 0$  since  $Z^{(\beta)}$  does not move after  $\tau$ . It also implies that  $\sigma^{(\beta)}$  can be written

$$\sigma^{(\beta)} = \liminf_{n \rightarrow \infty} \left\{ t \geq 0 : \int_0^{t \wedge \tau} X_s^2 ds \geq n \right\}.$$

On the other hand, a calculation similar to the one in the first part of the proof yields

$$\frac{1}{L_{t \wedge \tau}} = \mathcal{E}((\beta - 1)X \cdot W^{(\beta)})_{t \wedge \tau}, \quad t < \sigma^{(\beta)},$$

Hence on the event  $\{\sigma^{(\beta)} < \tau\}$ , we have  $\lim_{t \uparrow \sigma^{(\beta)}} L_t = \infty$  (up to a  $\mathbb{P}^{(\beta)}$ -nullset). But this is impossible since, by definition of  $\tau$ ,  $L_t$  cannot reach above  $1 + p(T)^{-1}$  prior to  $\tau$ . It follows that  $\mathbb{P}^{(\beta)}(\sigma^{(\beta)} < \tau) = 0$ , and consequently

$$\mathbb{P}^{(\beta)}(\sigma^{(\beta)} < \infty) = 0.$$

The proposition is proved.  $\square$

## 5.1 Example 1

Using Proposition 5.1 we can immediately construct an example of a price process  $S$  with nonnegative drift, which satisfies both NFLVR<sub>C</sub> and NUPBR, but not NA, on the time interval  $[0, T]$ . In other words, we have both an equivalent supermartingale measure and a local martingale deflator, but no local martingale measure (and certainly no martingale measure.) Indeed, in the notation of Proposition 5.1, set

$$S = \frac{1}{Z^{(1)}}.$$

Then

$$\frac{dS_t}{S_t} = X_t \mathbf{1}_{\{t \leq \tau\}} dW_t + X_t^2 \mathbf{1}_{\{t \leq \tau\}} dt,$$

so that  $S$  is strictly positive with positive drift. The only candidate density process is  $Z^{(1)}$ , which is a local martingale deflator but fails to be a true martingale. Hence  $\mathcal{D}_{\text{loc}} \neq \emptyset$  but  $\mathcal{M}_{\text{loc}} = \emptyset$ . On the other hand,  $Z^{(\beta)}$  is a true martingale and thus induces a measure  $\mathbb{P}^{(\beta)}$  that is equivalent to  $\mathbb{P}$  on  $\mathcal{F}_T$ . Since

$$\frac{dS_t}{S_t} = X_t \mathbf{1}_{\{t \leq \tau\}} dW_t^{(\beta)} - (\beta - 1) X_t^2 \mathbf{1}_{\{t \leq \tau\}} dt,$$

it follows that  $S$  is a supermartingale under  $\mathbb{P}^{(\beta)}$ , which thus lies in  $\mathcal{M}_{\text{sup}}$ .

Note that  $Z^{(1)} \in \mathcal{D}_{\text{loc}}$  has the property that  $Z^{(1)} S = 1$  is a true martingale. Hence by Theorem 4.1 the price process  $S$  is supported by some constrained representative agent equilibrium, and ND<sub>C</sub> holds.

## 5.2 Example 2

In Theorem 4.1, the representative agent has stochastic utility in general. We now modify the previous example slightly to get a price process  $S$  satisfying NFLVR<sub>C</sub> and NUPBR, but not NA, with the property that the constant strategy  $\hat{H} \equiv 1$  is optimal for an agent with logarithmic utility  $U(x) = \log x$ . In other words, in equilibrium the investor does not short the risky asset. This log-utility agent is therefore a valid representative agent for an equilibrium supporting the price process  $S$ . The key for this is to make the drift of  $S$  strictly

positive for all  $t \in [0, T]$ , not just prior to  $\tau$ .<sup>2</sup> We define the candidate local martingale deflator by

$$Z = \mathcal{E} \left( -(X \mathbf{1}_{[0, \tau]} + \mathbf{1}_{[\tau, T]}) \cdot W \right),$$

where  $X$  solves  $dX_t = -X_t^2 dW_t$ . The candidate density process for an equivalent supermartingale measure is given by

$$Z^{\text{sup}} = \mathcal{E} \left( -(\beta X \mathbf{1}_{[0, \tau]} + \mathbf{1}_{[\tau, T]}) \cdot W \right)$$

for some  $\beta > 1$ . Finally, the price process is defined by

$$S = \frac{1}{Z}.$$

**Lemma 5.1.** *The process  $Z$  is a strict local martingale, and  $Z^{\text{sup}}$  is a true martingale.*

*Proof.* We would like to apply Proposition 5.1, and for this we first change probability measure to  $\tilde{\mathbb{P}}$  given by  $d\tilde{\mathbb{P}} = \tilde{Z}_T d\mathbb{P}$ , where  $\tilde{Z} = \mathcal{E}(\mathbf{1}_{[\tau, T]} \cdot W)$  is a true martingale by Novikov's condition. The process  $d\tilde{W}_t = dW_t - \mathbf{1}_{[\tau, T]}(t)dt$  is then Brownian motion under  $\tilde{\mathbb{P}}$ . Define  $\tilde{X}$  as the solution to  $d\tilde{X}_t = -\tilde{X}_t^2 d\tilde{W}_t$ , let  $\tilde{L}_t = \mathcal{E}(-\beta \tilde{X} \cdot \tilde{W}) / \mathcal{E}(-\tilde{X} \cdot \tilde{W})$ , and consider the stopping time  $\tilde{\tau} = \inf\{t \geq 0 : \tilde{L}_t \geq 1 + p(T)^{-1}\}$ . Then, with

$$\tilde{Z}^{(1)} = \mathcal{E} \left( -\tilde{X} \mathbf{1}_{[0, \tilde{\tau}]} \cdot \tilde{W} \right) \quad \text{and} \quad \tilde{Z}^{(\beta)} = \mathcal{E} \left( -\beta \tilde{X} \mathbf{1}_{[0, \tilde{\tau}]} \cdot \tilde{W} \right),$$

Proposition 5.1 shows that  $\tilde{Z}^{(1)}$  is strict local martingale and  $\tilde{Z}^{(\beta)}$  is a true martingale. This carries over to  $\tilde{Z}\tilde{Z}^{(1)}$  and  $\tilde{Z}\tilde{Z}^{(\beta)}$ , respectively. Now, by pathwise uniqueness of the defining SDE for  $\tilde{X}$  and  $X$ , together with the fact that  $\tilde{W}_t = W_t$  for  $t < \tau$ , it follows that  $\tilde{X}_t = X_t$  for  $t < \tau$ . Hence, for  $t < \tau$ , we have  $\tilde{L}_t = L_t$  and thus  $\tilde{\tau} = \tau$ . Consequently,  $\tilde{Z}\tilde{Z}^{(1)} = Z$  and  $\tilde{Z}\tilde{Z}^{(\beta)} = Z^{\text{sup}}$ , and this proves the lemma.  $\square$

As in the previous example it is now straightforward to verify that  $S$  has strictly positive drift,  $\mathcal{D}_{\text{loc}} = \{Z\}$ ,  $Z^{\text{sup}} \in \mathcal{D}_{\text{sup}}$ , and that  $\mathcal{M}_{\text{loc}} = \emptyset$  since  $Z$  is not a true martingale. It thus remains to show that the constant strategy  $\hat{H} \equiv 1$  is optimal under logarithmic utility. But this follows directly from the definition of  $S$ , Jensen's inequality, and the fact that  $Z$  is a local martingale deflator. Indeed, if  $H$  is any 1-admissible strategy and setting  $X_T = 1 + (H \cdot S)_T$ , we have

$$\mathbb{E}[\log X_T - \log S_T] = \mathbb{E}[\log(Z_T X_T)] \leq \log(\mathbb{E}[Z_T X_T]) \leq 0.$$

Note that the nonnegativity of  $H$  was not used. This means that  $\hat{H}$  is optimal also among unconstrained, 1-admissible strategies. However, since NA fails, it is not optimal among *all* admissible strategies. In this sense, the short sale constraint is binding, because an

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<sup>2</sup>Indeed, if  $S$  is a supermartingale on  $[\tau, T]$ , then  $\mathbb{E}[\log S_T] = \mathbb{E}[\log(S_T/S_{T \wedge \tau})] + \mathbb{E}[\log S_{T \wedge \tau}] \leq \mathbb{E}[\log S_{T \wedge \tau}]$ , where Jensen's inequality and the assumption  $\mathbb{E}[S_T \mid \mathcal{F}_{T \wedge \tau}] \leq S_{T \wedge \tau}$  were used. Hence it is always better to stop trading at  $\tau$ . Economically this is because a risk-averse agent will not take on the risk inherent in holding  $S$ , without being compensated by positive returns in excess of the risk-free rate.



unconstrained agent with access to any admissible strategy could produce arbitrage and therefore (dramatically) improve upon  $\hat{H}$ . Any such strategy would involve short positions. On the other hand, since the optimal strategy  $\hat{H}$  is strictly positive, it is tempting to think that the short sale constraint is not binding. As the above shows, this interpretation is misleading.

Another way of saying the same thing is the following: naively one may conjecture that a strictly positive strategy could be improved upon without violating the short sale constraint by adding “a little bit” of an (unconstrained) arbitrage strategy. This would imply that NA is incompatible with strictly positive optimal strategies. Of course the above shows that this is not the case—but what is interesting to note is that the analogous naive argument is correct for NUPBR; this (together with market clearing) is what drives Theorem 3.1.

This discussion suggests that optimal strategies need not change in a continuous way with respect to changes in investment constraints. Further investigation of this phenomenon is left for future research.

## 6 Conclusion

In this paper we have considered constrained informational efficiency of a financial market. That is, we study conditions under which a given price process can be viewed as the outcome of some equilibrium where agents are not allowed to short the risky asset. We find, somewhat surprisingly, that constrained informational efficiency implies the existence of a local martingale deflator. The explanation for this result resides with the role of market clearing: at any point in time, at least one agent is locally unconstrained. However, this intuition does not carry over to arbitrage opportunities—indeed, we give an example of an equilibrium with a single representative agent with logarithmic utility, optimally holds the full supply of the risky asset, but where arbitrage is nonetheless possible using unconstrained strategies. Less surprisingly, NFLVR<sub>C</sub> necessarily holds in equilibrium, and ND<sub>C</sub> holds whenever a representative agent equilibrium exists. The existence of such an equilibrium is fully characterized: we show that a constrained representative agent equilibrium exists if and only if NFLVR<sub>C</sub> holds and there is a local martingale deflator turning the price process into a true martingale. Moreover, in a multi-agent complete market equilibrium, these conditions are satisfied whenever all agents have strictly positive holdings in the risky asset.

The above shows that the inclusion of short sale constraints, arguably a fairly modest modification of the fully frictionless framework, yields a surprisingly rich set of phenomena. Nonetheless, much remains to be done. In the future it would be interesting to investigate more general trading constraints, multiple assets, and price processes with jumps. We end by listing some open questions that naturally arose from the analysis presented in the present paper. It would be enlightening to see these issues resolved.

- (i) If  $H, K$  are two C-maximal strategies, is the sum  $H + K$  again C-maximal? Since each investor strategy  $\hat{H}^k$  is C-maximal in equilibrium by Lemma 3.1, an affirmative answer

to this question would imply that  $\hat{H}^1 + \dots + \hat{H}^n = 1$  is again C-maximal, so that  $\text{ND}_C$  always holds in equilibrium.

- (ii) Is there a constrained informationally efficient market that is not consistent with any constrained representative agent equilibrium? Equivalently (see Theorem 4.1), does there exist a constrained equilibrium  $(S; U_k, x_k, \hat{H}^k : k = 1, \dots, n)$  such that  $ZS$  is a strict local martingale for every  $Z \in \mathcal{D}_{\text{loc}}$ ?
- (iii) If  $ZS$  is a true martingale for some  $Z \in \mathcal{D}_{\text{loc}}$ , then  $\text{ND}_C$  is satisfied. Does the converse hold? If not, is there a different dual characterization of  $\text{ND}_C$ ?

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